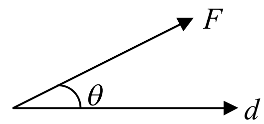
Work, Energy and Momentum

Work:
When a body moves a distance d along straight line, while acted on by a constant force of magnitude F in the same direction as the motion, the work done by the force is termed as

$$W = Fd$$

Now if the force F makes an angle θ with the direction of the displacement d then, the work done is



$$W = \vec{F} \cdot \vec{d} = FdCos\theta$$

Dimension: $[W] = [ML^2T^{-2}]$.
Unit: The SI unit of work is N-m or *Joule*.
Work is a scalar quantity and it can be both positive and negative. If the component of the force is in the same direction of the displacement, work is positive and otherwise negative.

Work done by a varying force:
Suppose a body of moves along a straight line. We divide the displacement into short segments, namely, Δx_1, Δx_2, Δx_3... etc and assume that an approximate force F_1 acts for Δx_1, F_2 for Δx_2 and so on, and then the work done is

$$W = F_1\Delta x_1 + F_2\Delta x_2 + F_3\Delta x_3 + \ldots\ldots\ldots$$

If the number of segment are very large and size of each is very small, then the work done

$$W = \int_{x_1}^{x_2} Fdx\,,$$ where x_1 and x_2 are initial and final positions of the body.

Potential energy:
This is the energy possessed by an object due to its state or position. If an object is raised from the ground to height then it gets gravitational potential energy due its position. If a spring is stretched then its gain potential energy due to its state.
Kinetic energy:
The energy due to the motion of an object is known as kinetic energy. I f the mass of an object is m and the velocity is v, then its kinetic energy would be

$$k = \frac{1}{2}mv^2$$

Gravitational potential energy:
When a gravitational force acts on a body undergoing a displacement, the force does work on the body, and as we shall see that this work can be expressed in terms of initial and final position of the body. If a body of mass m moves with vertically from a height y_1 to a height y_2 and the downward gravitational force on the body is its weight W, then the work done, in this case is represented as

$$W_g = -W(y_1 - y_2) = -mg(y_1 - y_2)$$

Thus W_g can be determined from the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight and the height above origin of the coordinates, is called the gravitational potential energy U.

$$U_{gravitational} = mgy$$

$$\therefore W_g = U_1 - U_2 = -\Delta U$$

Elastic potential energy:

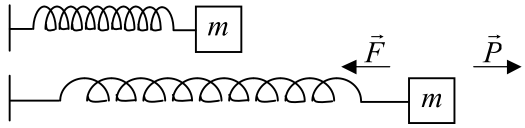


Figure 1: Elastic potential energy

Figure 1 shows a body of mass m on a level frictionless surface. One end of a spring is attached to the body and the other end of the spring is fixed. A force $\boldsymbol{P}$ is applied to stretch the spring. As soon as the slightest extension takes place, a force $\boldsymbol{F}$ is created within the spring which is opposite in direction to $\boldsymbol{P}$. The force $\boldsymbol{F}$ is called an elastic force. If the force $\boldsymbol{P}$ reduces to zero, the elastic force restores the spring to its original unstretched condition. It may therefore be referred to as a restoring force.

In a displacement from a initial elongation x_1 to a final elongation x_2 the elastic restoring force does an amount of work given by

$$W_{el} = \frac{1}{2}k{x_2}^2 - \left(-\frac{1}{2}k{x_1}^2\right)$$

The quantity $1/2kx^2$, one half the product of the force constant and the force constant and the square of the coordinate of the body, is called elastic potential energy of the body U.

$$U = \frac{1}{2}kx^2$$

Conservative and dissipative forces:
In the case of gravitational potential energy and elastic potential energy, we have seen that, the work done is independent of the path followed by the body and depends only upon the initial and final position. In both case the total mechanical energy remains constant and the force required for work done is a conservative force.

On the other hand, the frictional force is path-dependent. In this case the longer the path between two given points, the greater the work. When the friction acts alone, the total mechanical energy is not conserved. The friction is therefore called a non-conservative or dissipative force. The mechanical energy of a body is conserved only when no dissipative forces act on it.

Work energy theorem:
The work done on a body for a force is related very directly to the change in the body's motion that results. To develop the relationship we consider first a body of mass m moving along a straight line under the action of a constant resultant force of magnitude F directed along the line. Suppose the speed increase from v_1 to v_2, while the body undergoes a displacement s. Then we have

$$v_2^2 = v_1^2 + 2as$$

$$\therefore a = \frac{v_2^2 - v_1^2}{2s}$$

Hence the force F is

$$F = ma = m\left(\frac{v_2^2 - v_1^2}{2s}\right)$$

Then the work done is

$$W = Fs = m\left(\frac{v_2^2 - v_1^2}{2s}\right)s = \frac{m(v_2^2 - v_1^2)}{2}$$

$$\therefore W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The quantity $1/2mv^2$, one-half the product of the mass of the body and the square of it Its velocity is called kinetic energy k, i.e.

$$k = \frac{1}{2}mv^2$$

$$\therefore W = k_2 - k_1 = \Delta k$$

Thus *the work done on a body is always equal to the change in the kinetic energy of the body.* This statement is known as the **work energy theorem.**

If the work is positive the final kinetic energy is greater than the initial kinetic energy and the kinetic energy increases. If the work done is negative, the kinetic energy decreases. In the special case in which the work is zero, the kinetic energy remains constant.

Power:

Time considerations are not involved in the definition of work. It is important to consider the rate of the work. The rate at which work is done by a working agent is called the power developed by that agent.

If a quantity of work ΔW is done in a time interval Δt then the average power is defined as

$$Average\ power = \frac{Work\ done}{Time\ interval}$$

$$\therefore \overline{P} = \frac{\Delta W}{\Delta t}$$

The instantaneous power is defined as

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

Dimension: $[P] = [ML^2T^{-3}]$.

Unit: The SI unit of power is Js^{-1} or $Watt$.

Power and velocity:

Suppose a force $\boldsymbol{F}$ acts on a body while it undergoes a displacement Δs along its path. Then the power can be obtained from the equation

$$\overline{P} = \frac{\Delta W}{\Delta t} = \frac{F\Delta s}{\Delta t} = F\overline{v}$$

In the limit $\Delta t \to 0$,

$$P = Fv$$

Law of conservation of energy:

The law of conservation of energy state that, *energy can neither be created nor can destroyed, it can only be changed from one type to another and the total energy in the universe is constant.*

Momentum:

The product of the mass and velocity of an object is called the momentum. It is also sometimes called linear momentum and symbolized by p.

Therefore, $p = mv$.

Unit: The SI unit of momentum is $kgms^{-1}$.

Conservation of momentum:

The principle of conservation of momentum is one of the most fundamental and important principles of mechanics. It states that, *"When no external forces acts on a system, the total momentum of the system remains constant in magnitude and direction."*

Let us suppose that two masses of m_1 and m_2 are moving toward each other with initial velocities v_{01} and v_{02}. After collision, if the bodies move with velocities v_1 and v_2, then according to conservation of momentum

$$m_1v_{01} + m_2v_{02} = m_1v_1 + m_2v_2$$

Elastic collision:
If the total energy before and after collision are the same then this collision is called the elastic collision.
Let us suppose that two masses of m_1 and m_2 are moving toward each other with initial velocities v_{01} and v_{02}. After collision, if the bodies move with velocities v_1 and v_2, then

K.E before collision, $\frac{1}{2}m_1{v_{01}}^2 + \frac{1}{2}m_2{v_{02}}^2$

K.E aftercollision, $\frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$

Since the collision is elastic

$$\frac{1}{2}m_1{v_{01}}^2 + \frac{1}{2}m_2{v_{02}}^2 = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$

Center of mass:
The center of mass of a system of particles is a specific point at which, for many purposes, the system's mass behaves as if it were concentrated. The center of mass is a function only of the positions and masses of the particles that comprise the system.
In the case of a rigid body, the position of its center of mass is fixed in relation to the object. In the case of a loose distribution of masses in free space, such as, say, shot from a shotgun, the position of the center of mass is a point in space among them that may not correspond to the position of any individual mass. In the context of an entirely uniform gravitational field, the center of mass is often called the *center of gravity* — the point where gravity can be said to act.
Let us consider a collection of any number of particles whose total momentum P can defined as the vector sum of the individual momentum, as

$$P = p_1 + p_2 + p_3 + \ldots\ldots = m_1v_1 + m_2v_2 + m_3v_3 + \ldots\ldots$$

Now, if the velocity of the center of mass is V and the total mass of the body is M, then the total momentum can also be written as

$$P = MV = m_1v_1 + m_2v_2 + m_3v_3 + \ldots..$$

$$\therefore V = \frac{m_1v_1 + m_2v_2 + m_3v_3 + \ldots..}{m_1 + m_2 + m_3 + \ldots..}$$

This defines the velocity of the center of mass, and it also defines the position of the center of mass. Since the particle velocities are the time derivatives of the position vectors $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, $\boldsymbol{r_3}$ . . . of the particles, we define position vector of the center of mass, $\boldsymbol{R}$ as

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots..}{m_1 + m_2 + m_3 + \ldots..}$$

Exercise 6-10: A body of mass $2kg$ is initially at rest on a horizontal surface. If a horizontal force of $25N$ pushes it $4m$, using work energy relationship, find the velocity.

Solution:
The work-energy relationship is

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

We also know that $W = Fs$
Here, $v_1 = 0$, $F=25N$, $s = 4m$, $m=2kg$

$$\therefore (25N)(4m) = \frac{1}{2}(2kg){v_2}^2 - 2kg$$

$$\therefore v_2 = 10ms^{-1}$$

Exercise 6-12: A body of mass $8kg$ moves in a straight line on horizontal surface. At one point in its path its speed is $4ms^{-1}$. After it has traveled $3m$ its speed is $5ms^{-1}$ in the same direction. Use work-energy relationship and find the force acting on the body.

Solution:
Here, $v_1 = 4ms^{-1}, v_2 = 5ms^{-1}, m = 8kg, s = 3m$
From the work-energy relationship

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\Rightarrow Fs = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\Rightarrow F = \frac{\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}{s}$$

$$\Rightarrow F = \frac{\frac{1}{2}(8kg)(5ms^{-1})^2 - \frac{1}{2}(8kg)(4ms^{-1})^2}{3m} = \frac{100-64}{3}N = \frac{36}{3}N$$

$$\therefore F = 12N$$

Exercise 6-14: What is the potential energy of an $800kg$ elevator at the top of a building $380m$ above street level? Assume in the street level the potential energy is zero.

Solution:
We know, the potential energy is

$$U = mgy$$

Here $m = 800kg, y = 380m$

$$\therefore U = (800kg)(9.8ms^{-2})(380m) = 2.7 \times 10^6 J$$

Exercise 6-50: The hammer of a pile driver has a mass of $500kg$ and must be lifted a vertical distance of $2m$ in $3s$. What horse power engine is required?

Solution:
We know,

$$P = \frac{dW}{dt} = \frac{Fds}{dt} = \frac{mgds}{dt}$$

Here, $m = 500kg, ds = 2m, dt = 3s$

$$\therefore P = \frac{(500kg)(9.8ms^{-1})(2m)}{3s} = 3266.67Js^{-1} = 3266.67Watt$$

Now,

$$\because 746Watt = 1H.P.$$

$$\therefore P = \frac{3266.67}{746}H.P = 4.38H.P.$$

Problems for practice: Exercise 7-1, 7-3(a) (b) (c), 7-11(a) (b), 7-20(a) (b), 7-21, etc.